

On Borel Reducibility in Generalised Baire Space

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Abstract

In this paper we study the Borel reducibility of Borel equivalence relations on the generalised Baire space κ^κ for an uncountable κ with the property $\kappa^{<\kappa} = \kappa$. The theory looks quite different from its classical counterpart where $\kappa = \omega$, although some basic theorems do generalise.

We study the generalisations of classical descriptive set theory of Polish spaces to the setting where instead of the Baire space ω^ω we look at the generalised Baire space κ^κ of all functions from κ to κ where κ is an uncountable cardinal which satisfies $\kappa^{<\kappa} = \kappa$. The topology on this space is generated by the basic open sets

$$[p] = \{\eta \in \kappa^\kappa \mid \eta \supset p\}$$

where $p \in \kappa^{<\kappa}$. The resulting collection of open sets is closed under intersections of length $< \kappa$. The class of κ -Borel sets in this space is the smallest class containing the basic open sets and which is closed under taking unions and intersections of length κ .

In this paper we often work with spaces of the form $(2^\alpha)^\beta$ for some ordinals $\alpha, \beta \leq \kappa$. If $x \in (2^\alpha)^\beta$, then technically x is a function $\beta \rightarrow 2^\alpha$ and we denote by $x_\gamma = x(\gamma)$ the value at $\gamma < \beta$. Thus x_γ is a function $\alpha \rightarrow 2$ for each γ and we denote the value at $\delta < \alpha$ by $x_\gamma(\delta)$. The lengthier notation for $x \in (2^\alpha)^\beta$ is $(x_\gamma)_{\gamma < \beta}$ as a β -sequence of functions $\alpha \rightarrow 2$.

We say that a topological space is κ -Baire, if the intersection of κ many dense open sets is never empty. The generalised Baire space is κ -Baire [MV93]. If X is a topological space, we say that $A \subseteq X$ is κ -meager if its complement contains an intersection of κ many dense open sets. Thus, X is κ -Baire if and

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only if X is itself not meager (we always drop the prefix “ κ -”). The complement of a meager set is called *co-meager*. A set $A \subseteq X$ has the *Baire property* if there exists an open set U such that the symmetric difference $U \triangle A$ is meager. When we write $\forall^* x \in A(P(x))$ we mean that there is a co-meager set such that every element of that set which belongs to A satisfies P . We write $\exists^* x \in A(P(x))$ to mean that there exists a non-meager set such that every element of that set which belongs to A satisfies P .

1 Equivalence Relations Induced by a Group Action

Suppose G is a topological group. Let X be a Borel subset of κ^κ . An action $\rho: G \times X \rightarrow X$ is *Borel* if it is Borel as a function i.e. inverse images of open sets are Borel. This action induces an equivalence relation on X in which two elements x and y are equivalent if there exists $g \in G$ such that $\rho(g, x) = y$. For example, if the action is Borel, then it is easy to see that this equivalence relation is Σ_1^1 . This equivalence relation is denoted by $E_{G,\rho}^X$, or just E_G^X if the action is clear from the context.

Here are some examples of equivalence relations induced by a Borel action:

- id the identity relation.
- id^+ the jump of identity. This is an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha < \kappa}$ and $(y_\alpha)_{\alpha < \kappa}$ are equivalent if the sets $\{x_\alpha \mid \alpha < \kappa\}$ and $\{y_\alpha \mid \alpha < \kappa\}$ are equal (see Definition 5). This is not defined as an equivalence relation induced by a Borel action, but is easily seen to be bireducible with id_*^+ which is an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha < \kappa}$ and $(y_\alpha)_{\alpha < \kappa}$ are equivalent if there exists a permutation $s \in S_\kappa$ (S_κ is the group of all permutations of κ) such that $x_\alpha = y_{s(\alpha)}$ for all α . The latter is induced by a Borel action of S_κ .
- E_0 an equivalence relation on 2^κ where $(\eta, \xi) \in E_0$ if there exists $\alpha < \kappa$ such that for all $\beta > \alpha$ we have $\eta(\beta) = \xi(\beta)$.
- E_1 an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha < \kappa}$ and $(y_\alpha)_{\alpha < \kappa}$ are equivalent if there exists $\alpha < \kappa$ such that for all $\beta > \alpha$ we have $x_\beta = y_\beta$.

Since all the topologies in this paper are closed under intersections of length $< \kappa$, we replace “finite” by “less than κ ” when referring to product topologies below.

1 Theorem. *Let G be a discrete group of cardinality $\leq \kappa$ and let it act in a Borel way on a Borel subset $X \subseteq 2^\kappa$. Let E_G^X be the (Borel) equivalence relation induced by this action. Then $E_G^X \leq_B E_0$.*

Proof. The group G acts on $\mathcal{P}(G)^\kappa$ coordinatewise by multiplication on the right $g \cdot (X_i)_{i < \kappa} = (X_i g)_{i < \kappa}$. This gives rise to the equivalence relation $E_G^{P(G)^\kappa}$.

1.1 Claim. $E_G^X \leqslant_B E_G^{P(G)^\kappa}$.

Proof. Let $\pi: \kappa \rightarrow 2^{<\kappa}$ be a bijection. Let $x \in X$ and for each $\alpha < \kappa$ let

$$Z_\alpha(x) = \{g \in G \mid gx \in [\pi(\alpha)]\}.$$

This defines a reduction: an element $x \in X$ is mapped to $(Z_\alpha(x))_{\alpha < \kappa}$. Suppose there is $g_0 \in G$ such that $y = g_0 x$ for some $x, y \in X$. Then

$$\begin{aligned} Z_\alpha(x) &= \{g \in G \mid gx \in [\pi(\alpha)]\} \\ &= \{gg_0 \in G \mid gy \in [\pi(\alpha)]\} \\ &= Z_\alpha(y)g_0. \end{aligned}$$

On the other hand suppose that there exists $g \in G$ such that $Z_\alpha(x) = Z_\alpha(y)g$ for all $\alpha < \kappa$. It is enough to show that $g^{-1}y \in [p]$ for all basic open neighbourhoods $[p]$ of x . So suppose $U = [p]$ is a basic neighbourhood containing x and let $\alpha = \pi^{-1}(p)$. Now obviously $1_G \in Z_\alpha(x)$, so $1_G \in Z_\alpha(y)g$ and thus $g^{-1} \in Z_\alpha(y)$, i.e. $g^{-1}y \in [p]$. \square Claim 1.1

For a set S , F_S is the free group generated by elements of S . $F_\emptyset = F_0$ is the trivial group.

1.2 Claim. $E_G^{P(G)^\kappa} \leqslant_B E_{F_\kappa}^{P(F_\kappa)^\kappa}$.

Proof. Since G has size $\leqslant \kappa$ and F_κ is a free group on κ generators, there is a normal subgroup $N \subseteq F_\kappa$ such that $G \cong F_\kappa/N$. Assume without loss of generality that $G = F_\kappa/N$. Let pr be the canonical projection map $F_\kappa \rightarrow F_\kappa/N$. For $(A_\alpha)_{\alpha < \kappa} \in \mathcal{P}(G)^\kappa$, let

$$F((A_\alpha)_{\alpha < \kappa}) = (\text{pr}^{-1} A_\alpha)_{\alpha < \kappa}.$$

This is clearly a continuous reduction. \square Claim 1.2

1.3 Claim. $E_{F_\kappa}^{P(F_\kappa)^\kappa} \leqslant_B E_0$.

Proof. The space $\mathcal{P}(F_\kappa)^\kappa$ can be canonically thought to be the same as $(2^\kappa)^{F_\kappa}$ the bijection being defined by $(A_\alpha)_{\alpha < \kappa} \mapsto x$ where $x(g)(\alpha) = 1$ if and only if $g \in A_\alpha$. This space is equipped with the product topology (recall that our

definition of product topology is non-standard, see page 2): a basic open set is given by

$$\{x \mid x(g) \upharpoonright \alpha = p\}$$

for some ordinal $\alpha < \kappa$, some $g \in F_\kappa$ and some $p \in 2^{<\kappa}$ and the resulting collection of open sets is closed under intersections of length $< \kappa$. The action of F_κ on $(2^\kappa)^{F_\kappa}$ is then defined by $g * x = y$ where $y(f)(\alpha) = 1$ if and only if $x(fg^{-1})(\alpha) = 1$ for all $f \in F_\kappa$.

This space is more convenient for us to work with. Additionally we identify 2^α with $\mathcal{P}(\alpha)$ for all α and write $2^\alpha \subseteq 2^\beta$, meaning that an element p of 2^α is identified with q in 2^β where q is just p with “ $\beta - \alpha$ ” zeros at the end.

Let us look at the sets $X_\alpha = (2^\alpha)^{F_\alpha}$ for $\alpha \leq \kappa$.

If $w \in X_\beta$ and $\alpha < \beta$, denote by $w \upharpoonright \alpha$ the element v of X_α such that $v(g) = w(g) \upharpoonright \alpha$ for all $g \in F_\alpha$. Thus, by the identifications we made, $X_\alpha \subseteq X_\beta \subseteq X_\kappa$ for all $\alpha < \beta < \kappa$. For every $\alpha < \kappa$ fix a well-ordering $<_\alpha$ of X_α . Given $g \in F_\alpha$ and $w \in X_\alpha$, let $g * w \in X_\alpha$ be the element such that $(g * w)(f) = w(fg^{-1})$ for all $f \in F_\alpha$. This is an action of F_α on X_α and for $\alpha = \kappa$ it coincides with the original action of F_κ on $\mathcal{P}(F_\kappa)^\kappa$ under the mentioned identifications.

Fix $x \in X_\kappa$. For each α , let $x(\alpha)$ be the $<_\alpha$ -least element of

$$\{g * (x \upharpoonright \alpha) \mid g \in F_\alpha\}$$

and let $H(x) = (x(\alpha))_{\alpha < \kappa}$. We claim that for all $x, y \in X_\kappa$, $y = g * x$ for some $g \in F_\kappa$ if and only if there exists $\beta < \kappa$ such that for all $\alpha > \beta$, $x(\alpha) = y(\alpha)$.

Assume first that such $g \in F_\kappa$ exists. Then $g \in F_\beta$ for some $\beta < \kappa$ and for all $\alpha > \beta$, $g \in F_\alpha$. Thus, it is obvious that $x(\alpha) = y(\alpha)$ for $\alpha > \beta$, because for these α

$$\{g * (x \upharpoonright \alpha) \mid g \in F_\alpha\} = \{g * (y \upharpoonright \alpha) \mid g \in F_\alpha\}.$$

Assume now that there exists $\beta < \kappa$ such that for all $\alpha > \beta$, $x(\alpha) = y(\alpha)$. Then for each $\alpha > \beta$ there exists $g_\alpha \in F_\alpha$ such that $x \upharpoonright \alpha = g_\alpha * (y \upharpoonright \alpha)$. For each $\alpha > \beta$, let $\gamma(\alpha)$ be the least ordinal such that $g_\alpha \in F_{\gamma(\alpha)}$. If α is a limit ordinal, then $\gamma(\alpha) < \alpha$ and so there is γ_0 and a stationary $S_0 \subseteq \lim \kappa$ such that for all $\alpha \in S_0$ we have $g_\alpha \in F_{\gamma_0}$. Since $|F_{\gamma_0}| < \kappa$, there is a stationary $S \subseteq S_0$ and $g_* \in F_{\gamma_0}$ such that for all $\alpha \in S$ we have $g_\alpha = g_*$. Since S is unbounded, this obviously implies that $y = g_* * x$.

Fix bijections $f_\alpha: X_\alpha \rightarrow \kappa$ and map each $x \in X_\kappa$ to the sequence $(f_\alpha(x(\alpha)))_{\alpha < \kappa}$ and denote this mapping by G . By the above we have $x = g * y$ for some $g \in F_\kappa$ if and only if $(G(x), G(y)) \in E_0$. It remains to show that G is continuous.

Suppose $x \in X_\kappa$ and take an open neighbourhood U of $G(x)$. Then there is β such that

$$\{\eta \in \kappa^\kappa \mid \forall \alpha < \beta (\eta(\alpha) = f_\alpha(x(\alpha)))\} \subseteq U.$$

Now, the set $\{y \in F_\kappa \mid y \restriction \beta = x \restriction \beta\}$ is mapped inside U and contains x , so it remains to show that this set is open, but this follows from the definition of the topology on $(2^\kappa)^{F_\kappa}$ in particular that the collection of open sets is closed under intersections of length $< \kappa$. □ Claim 1.3

□ Theorem 1

2 Theorem ($V = L$). *There is a Borel equivalence relation E whose classes have size 2, which is smooth (i.e. Borel reducible to id) yet which is not induced by a Borel action of a group of size $\leq \kappa$.*

Proof.

2.1 Claim. *There is an open dense set $O \subseteq 2^\kappa$ and a bijection $f: O \rightarrow 2^\kappa \setminus O$ such that the graph of f is Borel, but f is not Borel as a function on any non-meager Borel set. However the inverse of f is Borel.*

Proof. We let O be the complement of a certain closed set of “master codes” for size κ initial segments of L . This is defined as follows. Let \mathcal{L} be the language of set theory augmented by constant symbols $\bar{\alpha}$ for each ordinal $\alpha < \kappa$. Also let T_0 denote the theory ZFC^- (ZFC minus the power set axiom) plus $V = L$ plus the statement “there are only boundedly many ordinals β such that L_β satisfies ZFC^- ”. We consider complete, consistent theories T which extend T_0 and which in addition satisfy the following:

1. There is no ω -sequence of formulas $\varphi_n(x)$ (mentioning constants $\bar{\alpha}$ for $\alpha < \kappa$) such that for each n both the sentence “ $\exists! x \varphi_n(x)$ ” and the sentence “ $\exists x, y (\varphi_n(x) \wedge \varphi_{n+1}(y) \wedge y \in x)$ ” belong to T .
2. For each $\beta < \kappa$ and formula $\varphi(x)$ (mentioning constants $\bar{\alpha}$ for $\alpha < \kappa$) if the sentences “ $\exists! x \varphi(x)$ ” and “ $\exists x (\varphi(x) \wedge x < \bar{\beta})$ ” both belong to T then so does the sentence “ $\exists x (\varphi(x) \wedge x = \bar{\gamma})$ ” for some $\gamma < \beta$.

By identifying sentences of \mathcal{L} with ordinals less than κ we can regard theories in \mathcal{L} as subsets of κ . Now let $C \subseteq 2^\kappa$ be the set of theories T as above. Then C is a closed set. And C is nowhere dense as any set of \mathcal{L} -sentences of size less than κ is included in an inconsistent such set.

The theories in C are exactly the first-order theories of structures of the form L_β in which the constant symbol $\bar{\alpha}$ is interpreted as the ordinal α for each $\alpha < \kappa$ and in which the axioms of T_0 hold. And for each T in C there is a unique such model $M(T)$ in which every element is definable from parameters less than κ .

We are ready to define the function $f: O \rightarrow C$ where O is the complement of C in 2^κ . List the elements of O in $<_L$ -increasing order as x_0, x_1, \dots and list the elements of C in $<_L$ -increasing order as y_0, y_1, \dots ; then we set $f(x_i) = y_i$

for each $i < \kappa^+$. The inverse of f is Borel because given $y \in C$ we can identify $f^{-1}(y)$ (viewed as a subset of κ) as the set of $\gamma < \kappa$ such that the sentence “ γ belongs to the i -th element in the $<_L$ -increasing enumeration of 2^κ where i is the order type of the set of β such that L_β models T_0 ” belongs to (the theory associated to) y . Thus the graph of f is Borel. If B is a non-meager Borel set and g is a Borel function then we claim that g cannot agree with f on B : Indeed, let β_0 be so that L_{β_0} models ZFC^- and contains Borel codes for both B and g and let $x \in B \cap O$ be κ -Cohen generic over L_{β_0} . Then $f(x) = T$ is the theory of a model $M(T) = L_\beta$ where β is greater than β_0 . But $g(x)$ belongs to $L_{\beta_0}[x]$, which by the genericity of x is a model of ZFC^- while $L_{\beta_0}[f(x)]$ does not satisfy ZFC^- as $f(x) = T$ codes the model L_β . \square

Define xEy if and only if $x = y$, $y = f(x)$ or $x = f(y)$. Now E has a Borel transversal, i.e., there is a Borel function t such that $xEt(x)$ for all x and xEy if and only if $t(x) = t(y)$ for all x, y : Given $x \in 2^\kappa$, first decide in a Borel way if x is in O or not. If yes, then let $t(x) = x$, otherwise, find $f^{-1}(x)$ in a Borel way (since f^{-1} is Borel) and let $t(x) = f^{-1}(x)$. This $t(x)$ is a Borel transversal. It follows that E is smooth.

Finally, suppose E is given by a Borel action of some group G of size at most κ . Then for each $x \in O$ choose $g_x \in G$ such that $f(x) = g_x \cdot x$; then for some fixed $g \in G$, $f(x) = g \cdot x$ for non-meager many $x \in O$, contradicting the fact that f is not Borel on any non-meager Borel set. \square

3 Question. Is there a Borel equivalence relation with classes of size κ which is not reducible to E_0 ?

2 E_1 and E_{club}

Let E_1 be the equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha < \kappa}$ and $(y_\alpha)_{\alpha < \kappa}$ are equivalent if there exists $\beta < \kappa$ such that for all $\gamma > \beta$, $x_\gamma = y_\gamma$.

4 Theorem. E_1 and E_0 are bireducible.

Proof. It is obvious that $E_0 \leq_B E_1$, so let us look at the other direction.

To simplify notation, we think of E_0 on κ^κ : two functions η and ξ are E_0 -equivalent if the set $\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\}$ is bounded. It is easy to see that E_0 on 2^κ is bireducible with this equivalence relation.

For all limit $\alpha < \kappa$, define E_1^α to be the equivalence relation on $(2^\alpha)^\alpha$ approximating E_1 i.e. $(x_i)_{i < \alpha} E_1^\alpha (y_i)_{i < \alpha}$ if for some $\beta < \alpha$, $x_i = y_i$ for all $i > \beta$. Now define the reduction $F: (2^\kappa)^\kappa \rightarrow \kappa^\kappa$ so that for all $(x_i)_{i < \kappa} \in (2^\kappa)^\kappa$,

$F((x_i)_{i<\kappa})(\alpha) = 0$ if α is not a limit and otherwise it is a code for the E_1^α -equivalence class of $(x_i \upharpoonright \alpha)_{i<\alpha}$.

Clearly F is continuous and if $(x_i)_{i<\kappa} E_1 (y_i)_{i<\kappa}$, then also $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are E_0 -equivalent (if $\beta < \kappa$ witnesses the first equivalence, it witnesses also the second).

Also if $(x_i)_{i<\kappa}$ and $(y_i)_{i<\kappa}$ are not E_1 equivalent, then for all $\alpha < \kappa$ there are $\gamma, \beta < \kappa$ such that $\beta > \alpha$ and $x_\beta(\gamma) \neq y_\beta(\gamma)$. Let $f(\alpha)$ be $\max\{\beta, \gamma\}$. Now if $\alpha^* < \kappa$ is such that for all $\alpha < \alpha^*$, $f(\alpha) < \alpha^*$, then clearly $(x_i \upharpoonright \alpha^*)_{i<\alpha^*}$ and $(y_i \upharpoonright \alpha^*)_{i<\alpha^*}$ are not $E_1^{\alpha^*}$ -equivalent, and thus $F((x_i)_{i<\kappa})(\alpha^*) \neq F((y_i)_{i<\kappa})(\alpha^*)$. Since the set of such α^* is unbounded, $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are not E_0 -equivalent. \square

5 Definition. If E is an equivalence relation on 2^κ , its *jump* is the equivalence relation denoted by E^+ on $(2^\kappa)^\kappa$ defined as follows. Two sequences $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are E^+ -equivalent, if

$$\{[x_\alpha]_E \mid \alpha < \kappa\} = \{[y_\alpha]_E \mid \alpha < \kappa\}$$

where $[x]_E$ is the equivalence class of x in E . Since $(2^\kappa)^\kappa$ is homeomorphic to 2^κ we can assume without loss of generality that E^+ is also defined on 2^κ .

For an ordinal $\alpha < \kappa^+$ define $E^{\alpha+}$ by transfinite induction. To begin, define $E^{0+} = E$. If $E^{\alpha+}$ is defined, then $E^{(\alpha+1)+} = (E^{\alpha+})^+$.

Suppose α is a limit and $E^{\beta+}$ is defined to be an equivalence relation on 2^κ for $\beta < \alpha$. Let X be the disjoint union of α many copies of 2^κ . Denote the β :th copy by X_β , thus $X = \bigcup_{\beta<\alpha} X_\beta$. Let h be a homeomorphism $X \rightarrow 2^\kappa$. Two functions η and ξ are defined to be $E^{\alpha+}$ -equivalent, if $h^{-1}(\eta)$ and $h^{-1}(\xi)$ belong both to the same X_β and are $E^{\beta+}$ -equivalent. This is called the *join* of the equivalence relations $\{E^{\beta+} \mid \beta < \alpha\}$ and is denoted $\bigoplus_{\beta<\alpha} E^{\beta+}$.

6 Theorem. $E_0 <_B \text{id}^+$

Proof. The reduction is defined by

$$E_0 \leqslant_B \text{id}^+ : \eta \mapsto (p + \eta)_{p \in 2^{<\kappa}}.$$

Suppose $f: 2^\kappa \rightarrow (2^\kappa)^\kappa$ is a Borel reduction from id^+ to E_0 . There is a co-meager set D on which f is continuous. Without loss of generality assume that this D is the intersection $\bigcap_{i<\kappa} D_i$ where D_i are dense open.

For every $i < \kappa$ we will define ordinals γ_i together with sequences $x^i = (x_\alpha^i)_{\alpha<\gamma_i}$ and $y^i = (y_\alpha^i)_{\alpha<\gamma_i}$ where each $x_\alpha^i, y_\alpha^i \in 2^{\gamma_i}$ and permutations $\pi_i \in S_{\gamma_i}$. These will satisfy the following requirements for every $i < j < \kappa$:

1. $\pi_i \subseteq \pi_j$.
2. $\gamma_i \leq \gamma_j$,
3. For all $\alpha < \gamma_i$ we have $x_\alpha^i \subseteq x_\alpha^j$ and $y_\alpha^i \subseteq y_\alpha^j$.
4. For all $\alpha < \gamma_i$ we have $x_\alpha^i = y_{\pi_i(\alpha)}^i$.
5. Let $[(x_\alpha^i)_{\alpha < \gamma_i}]$ be the set of all $x = (x_\alpha)_{\alpha < \kappa} \in (2^\kappa)^\kappa$ such that $x_\alpha^i \subseteq x_\alpha$ for all α . There exist $\beta > i$, $\delta < \kappa$ and $p, q \in (2^\delta)^{\beta+1}$ such that

$$f([(x_\alpha^i)_{\alpha < \gamma_i}] \cap D) \subseteq [p],$$

$$f([(y_\alpha^i)_{\alpha < \gamma_i}] \cap D) \subseteq [q]$$

and $p(\beta) \neq q(\beta)$.

6. $[(x_\alpha^{i+1})_{\alpha < \gamma_{i+1}}] \subseteq D_i$ and $[(y_\alpha^{i+1})_{\alpha < \gamma_{i+1}}] \subseteq D_i$

This will lead to a contradiction as follows. Let $\tilde{x} = (\tilde{x}_\alpha)_{\alpha < \kappa}$ be such that for every α we have $\tilde{x}_\alpha \upharpoonright \gamma_i = x_\alpha^i$ if $\gamma_i > \alpha$. This is possible by (2) and (3). Analogously define \tilde{y} . Now by (1) we can define $\pi = \bigcup_{i < \kappa} \pi_i$ which by (4) witnesses that \tilde{x} and \tilde{y} are id^+ -equivalent. By (6) they are in D and by continuity in D and by (5) the images $f(\tilde{x})$ and $f(\tilde{y})$ cannot be E_0 -equivalent.

Let $x^* = (x_\alpha^*)_{\alpha < \kappa}$ and $y^* = (y_\alpha^*)_{\alpha < \kappa}$ be any sequences in D such that x^* is not id^+ -equivalent to y^* . Find these for example as follows: We will define sequences $(\xi_k)_{k < \kappa}$ and $(\eta_k)_{k < \kappa}$ and ordinals ε_k such that for all $k < \kappa$ we have $\xi_k, \eta_k \in (2^{\varepsilon_k})^{\varepsilon_k}$, for $k_1 < k_2$ we have $\varepsilon_{k_1} < \varepsilon_{k_2}$, $\xi_{k_1} \subseteq \xi_{k_2}$ and $\eta_{k_1} \subseteq \eta_{k_2}$, and the unions $\bigcup_{k < \kappa} \xi_k$ and $\bigcup_{k < \kappa} \eta_k$ are in D and not id^+ -equivalent. This is easy: Let $\varepsilon_0 = 0$, $\xi_0 = \emptyset$ and $\eta_0 = \emptyset$. If ξ_k and η_k are defined, first extend ξ_k to an element $\xi'_{k+1} \in (2^{\varepsilon'_{k+1}})^{\varepsilon'_{k+1}}$ (for suitable $\varepsilon'_{k+1} > \varepsilon_k$) such that $[\xi'_{k+1}] \subseteq D_k$. Then extend the first component of η_k so that it differs in a diagonal way from every component of ξ'_{k+1} . After that, extend the result into $\eta_{k+1} \in (2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ (for suitable $\varepsilon_{k+1} > \varepsilon'_{k+1}$) so that $[\eta_{k+1}] \subseteq D_k$ and $\varepsilon_{k+1} > \varepsilon'_{k+1}$. Finally extend ξ'_{k+1} to an element of $(2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ so that the first component of η_{k+1} is still diagonally different from every component of ξ_{k+1} ; technically this means that $\eta_{k+1}(0)(\alpha) \neq \xi_{k+1}(\alpha)(\alpha)$. At limit k just take the natural limits of the sequences. In this way at the κ :th limit we obtain ξ_κ and η_κ are as required, so we can define $x^* = \xi_\kappa$ and $y^* = \eta_\kappa$.

Let β and δ be such that $f(x^*)(\beta)(\delta) \neq f(y^*)(\beta)(\delta)$ which exist because f is assumed to be a reduction and $f(x^*)$ and $f(y^*)$ are not E_0 -equivalent. Now by continuity in D there is $\gamma_0^* > 0$ such that

$$f([(x_\alpha^* \upharpoonright \gamma_0^*)_{\alpha < \gamma_0^*}] \cap D) \subseteq [(f(x^*) \upharpoonright (\beta + 1))]$$

and

$$f[(y_\alpha^* \upharpoonright \gamma_0^*)_{\alpha < \gamma_0^*} \cap D] \subseteq [(f(y^*) \upharpoonright (\beta + 1))]$$

Then we glue $(x_\alpha^* \upharpoonright \gamma_0^*)_{\alpha < \gamma_0^*}$ to the end of $(y_\alpha^* \upharpoonright \gamma_0^*)_{\alpha < \gamma_0^*}$ and vice versa:
Let $\gamma_0 = \gamma_0^* + \gamma_0^*$ and for all $\alpha < \gamma_0^*$ define

$$\begin{aligned} x_\alpha^0 &= x_\alpha^* \upharpoonright \gamma_0 \\ y_\alpha^0 &= y_\alpha^* \upharpoonright \gamma_0 \\ x_{\gamma_0^* + \alpha}^0 &= y_\alpha^* \upharpoonright \gamma_0 \\ y_{\gamma_0^* + \alpha}^0 &= x_\alpha^* \upharpoonright \gamma_0 \end{aligned}$$

Let π_0 be the permutation which takes α to $\gamma_0 + \alpha$ when $\alpha < \gamma_0$ and if $\alpha = \gamma_0 + \varepsilon$, then $\pi(\alpha) = \varepsilon$. So we have defined π_0 , γ_0 , $(x_\alpha^0)_{\alpha < \gamma_0}$ and $(y_\alpha^0)_{\alpha < \gamma_0}$ such that all the conditions (1)–(6) are satisfied so far.

Suppose that π_i , γ_i , $(x_\alpha^i)_{\alpha < \gamma_i}$ and $(y_\alpha^i)_{\alpha < \gamma_i}$ are defined for $i < j$ such that the conditions (1)–(6) are satisfied. If j is a limit, then just define $\pi_j = \bigcup_{i < j} \pi_i$, $x_\alpha^j = \bigcup_{i' < i < j} x_\alpha^{i'}$ and $y_\alpha^j = \bigcup_{i' < i < j} y_\alpha^{i'}$ for some i' such that $\gamma_{i'} > \alpha$ and $\gamma_j = \sup_{i < j} \gamma_i$ and $\gamma_j = \sup_{i < j} \gamma_i$.

Suppose j is a successor, in fact w.l.o.g denote the predecessor by i , i.e. $j = i + 1$. Next we want to build elements $x^* = (x_\alpha^*)_{\alpha < \kappa}$ in $[(x_i)_{i < \gamma_i}] \cap D$ and $y^* = (y_\alpha^*)_{\alpha < \kappa}$ in $[(y_i)_{i < \gamma_i}] \cap D$ such that $x_\alpha^* = y_{\pi(\alpha)}^*$ for all $\alpha < \gamma_i$ and which are not id^+ -equivalent. To do that, define $\varepsilon_0 = \gamma_i$, $\xi_0 = (x_\alpha^i)_{\alpha < \gamma_i}$ and $\eta_0 = (y_\alpha^i)_{\alpha < \gamma_i}$. Suppose we have defined ξ_k and η_k for some $k < \kappa$.

First we extend ξ_k to $\xi'_{k+1} \in (2^{\varepsilon'_{k+1}})^{\varepsilon'_{k+1}}$ for some suitable $\varepsilon'_{k+1} < \kappa$, $\varepsilon'_{k+1} > \varepsilon_k$ so that $[\xi'_{k+1}] \subseteq D_k$. Then we extend η_k first to a $\eta'_{k+1} \in (2^{\varepsilon'_{k+1}})^{\varepsilon'_{k+1}}$ such that $\eta'_{k+1} \upharpoonright \gamma_i$ equals to the action of π_i applied to $\xi'_{k+1} \upharpoonright \gamma_i$ and $\eta'_{k+1} \upharpoonright \{\gamma_i\}$ diagonally differs from every component of ξ'_{k+1} . Then extend η'_{k+1} to $\eta_{k+1} \in (2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ (for a suitable $\varepsilon_{k+1} > \varepsilon'_{k+1}$) so that $[\eta_{k+1}] \subseteq D_k$. Finally extend ξ'_{k+1} to $\xi_{k+1} \in (2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ in any such way that $\eta_{k+1} \upharpoonright \{\gamma_i\}$ differs from every component of ξ_{k+1} in a diagonal way.

At limit k just take the natural limits of the sequences. In this way at the κ :th limit we obtain ξ_κ and η_κ which are as required, so we can define $x^* = \xi_\kappa$ and $y^* = \eta_\kappa$.

Now by continuity and by the fact that x^* and y^* are not id^+ -equivalent, find γ_{i+1}^* and $\beta > i + 1$ so that $f(x^*)(\beta) \neq f(y^*)(\beta)$ and

$$f[(x_\alpha^* \upharpoonright \gamma_{i+1}^*)_{\alpha < \gamma_{i+1}^*} \cap D] \subseteq [(f(x^*) \upharpoonright (\beta + 1))]$$

and

$$f[(y_\alpha^* \upharpoonright \gamma_{i+1}^*)_{\alpha < \gamma_{i+1}^*} \cap D] \subseteq [(f(y^*) \upharpoonright (\beta + 1))].$$

Also we make sure that γ_{i+1}^* is big enough so that (6) is satisfied. Now we want to glue a part of $(x_\alpha^* \upharpoonright \gamma_{i+1}^*)_{\alpha < \gamma_{i+1}^*}$ to the end of $(y_\alpha^* \upharpoonright \gamma_{i+1}^*)_{\alpha < \gamma_{i+1}^*}$ and vice versa: Let $\varepsilon = \gamma_{i+1} - \gamma_i$, i.e. the order type of $\gamma_{i+1}^* \setminus \gamma_i$ and let $\gamma_{i+1} = \gamma_{i+1}^* + \varepsilon$. Define x_α^{i+1} and y_α^{i+1} for all $\alpha < \gamma_{i+1}$ depending on α as follows. If $\alpha < \gamma_{i+1}^*$, let x_α^{i+1} to be $x_\alpha^* \upharpoonright \gamma_{i+1}$ and y_α^{i+1} to be $y_\alpha^* \upharpoonright \gamma_{i+1}$. If $\alpha = \gamma_{i+1}^* + \delta$ for some $\delta < \varepsilon$, then let x_α^{i+1} to be $y_{\gamma_i + \delta}^*$ and y_α^{i+1} to be $x_{\gamma_i + \delta}^*$. This gives us also π_{i+1} and we are done. \square

7 Definition. For a regular cardinal $\mu < \kappa$ and $\lambda \in \{2, \kappa\}$ let $E_{\mu\text{-cub}}^\lambda$ be the equivalence relation on λ^κ such that η and ξ are $E_{\mu\text{-cub}}^\lambda$ -equivalent if the set $\{\alpha \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ -cub, i.e. an unbounded set which is closed under μ -cofinal limits. If T is a countable complete first-order theory, denote by \cong_κ^T the isomorphism relation on the models of T .

In the following we show that

1. The α :th jump of identity for $\alpha < \kappa^+$ is reducible to $E_{\mu\text{-cub}}^\kappa$ for every regular $\mu < \kappa$,
2. Every Borel isomorphism relation is reducible to $E_{\mu\text{-cub}}^\kappa$ for every regular $\mu < \kappa$,
3. If T a countable complete first-order classifiable (superstable with NDOP and NOTOP) and shallow theory, then $\cong_\kappa^T \leq_B E_{\mu\text{-cub}}^\kappa$.

8 Definition. Fix a limit ordinal $\alpha \leq \kappa$ and let t be a subtree of $2^{<\omega}$ with no infinite branches. Let h be a function from the leaves of t to $2^{<\alpha}$. Then (t, h) determines the set $B_{(t, h)}$ as follows: $p \in 2^\alpha$ belongs to $B_{(t, h)}$ if player **II** has a winning strategy in the game $G(p, t, h)$: The players start at the root and then one after another choose a successor of the node they are in and then move to that successor. Player **I** starts. Eventually they reach a leaf l and player **II** wins if $h(l) \subset p$. We say that (t, h) is a *Borel code* for α .

If $\alpha = \kappa$, it is easy to see by induction on the rank of the tree that $B_{(t, h)}$ is a usual Borel set and conversely, if $B \subset 2^\kappa$ is any Borel set, then there is a Borel code (t, h) for κ such that $B = B_{(t, h)}$.

If t is replaced by a more general $\kappa^+ \kappa$ -tree (subtree of $\kappa^{<\kappa}$ without branches of length κ), then the sets that are obtained in this way are the so called Borel* sets, see [Bla81, MV93, Hal96, FHK13].

Suppose (t, h) is a Borel code for κ and $\alpha < \kappa$. Say that α is *good* for (t, h) , if for all leaves $l \in t$ with $\text{ht}(l) < \alpha$ we have $h(l) \in 2^{<\alpha}$. Clearly the set of good α for a fixed (t, h) is a cub set.

Define the α :th *approximation* of (t, h) , denoted $(t, h) \restriction \alpha$ to be the pair $(t \restriction \alpha, h \restriction \alpha)$ where $t \restriction \alpha = t \cap \alpha^{<\omega}$ and for all leaves l of $t \restriction \alpha$, $(h \restriction \alpha) = h \restriction (t \restriction \alpha)$. It is obvious that if (t, h) is a Borel code for κ and $\alpha < \kappa$ is good for (t, h) , then $(t, h) \restriction \alpha$ is a Borel code for α .

By replacing $2^{<\alpha}$ by $(2^{<\alpha})^2$ for the range of h and making necessary changes we can define Borel codes for subsets of $(2^\alpha)^2$.

Note that the game $G(p, t, h)$ is determined for all $p \in 2^\alpha$ (this is not the case for general Borel*-sets).

Make a similar definition for codes of Borel subsets of $2^\kappa \times 2^\kappa$.

9 Lemma. *Suppose that $B = B_{(t,h)}$ is a Borel subset of $2^\kappa \times 2^\kappa$. Then*

$$(\eta, \xi) \in B \iff (\eta \restriction \alpha, \xi \restriction \alpha) \in B_{(t,h) \restriction \alpha}$$

for cub-many α and $(\eta, \xi) \notin B \iff (\eta \restriction \alpha, \xi \restriction \alpha) \notin B_{(t,h) \restriction \alpha}$ for cub-many α .

Proof. Suppose $(\eta, \xi) \in B$ and let σ be a winning strategy of player **II** in $G((\eta, \xi), t, h)$. Let C be the set of those limit α which are good for (t, h) and that $t \restriction \alpha$ is closed under σ . Clearly $(\eta \restriction \alpha, \xi \restriction \alpha) \in B_{(t,h) \restriction \alpha}$ for all $\alpha \in C$ and C is cub.

Conversely, if $(\eta, \xi) \notin B$, then player **I** has a winning strategy τ in $G((\eta, \xi), t, h)$ and by closing under τ we obtain the needed cub set again. \square

10 Lemma. *Let S be the set of Borel equivalence relations E such that for some Borel code (t, h) , $E = B_{(t,h)}$ and $B_{(t,h) \restriction \alpha}$ is an equivalence relation for cub-many $\alpha < \kappa$. Then S contains id and is closed under jump and the join operation \oplus as in the definition of iterated jump, Definition 5.*

Proof. Enumerate $2^{<\kappa} = \{p_\alpha \mid \alpha < \kappa\}$. Let $t = \kappa^1$ and $h(\alpha) = (p_\alpha, p_\alpha)$. Clearly $\text{id} = B_{(t,h)}$ and for those α for which $\{p_i \mid i < \alpha\} = 2^{<\alpha}$, $B_{(t,h) \restriction \alpha}$ is the identity on 2^α and this is clearly a cub set.

Suppose E is in S and (t, h) is a code for E witnessing that and that C is a cub set on which $E_{(t,h)}$ is an equivalence relation. It is not difficult to design a Borel code (t^+, h^+) for the jump E^+ and check that for cub many $\alpha \in C$, $B_{(t^+, h^+) \restriction \alpha}$ is the jump of $B_{(t,h) \restriction \alpha}$.

Similarly suppose that $E_i \in S$ are equivalence relations for $i < \kappa$ and witnessing codes (t_i, h_i) are given with C_i cub sets such that $B_{(t_i, h_i) \restriction \alpha}$ is an equivalence relation for each $\alpha \in C_i$. Then it is not difficult to design a code (t, h) so that $B_{(t,h)}$ is $\bigoplus_{i < \kappa} E_i$ and for cub many $\alpha \in \bigcap_{i < \kappa} C_i$, $B_{(t,h) \restriction \alpha}$ is $\bigoplus_{i < \alpha} B_{(t_i, h_i) \restriction \alpha}$ \square

It follows that S contains all iterates of the jump $\text{id}^{+\beta}$, $\beta < \kappa^+$.

11 Theorem. *Let E be an equivalence relation in S . Then E is reducible to $E_{\mu\text{-cub}}^\kappa$ for any regular $\mu < \kappa$ (see Definition 7).*

Proof. Let E be $B_{(t,h)}$ where (t, h) witnesses that E belongs to S . To each η assign the function f_η where $f_\eta(\alpha)$ is a code for the $B_{(t,h)\upharpoonright\alpha}$ equivalence class of $\eta \upharpoonright \alpha$ (if $\text{cf}(\alpha) = \mu$ and $B_{(t,h)\upharpoonright\alpha}$ is an equivalence relation, 0 otherwise). By Lemma 9, if $\eta E \xi$ then $f_\eta(\alpha) = f_\xi(\alpha)$ for μ -cub-many α and if $\neg \eta E \xi$ then $f_\eta(\alpha) \neq f_\xi(\alpha)$ for μ -cub-many α . \square

12 Corollary. *The iterated jumps $\text{id}^{\alpha+}$ of the identity are reducible to $E_{\mu\text{-cub}}^\kappa$ for each regular $\mu < \kappa$.* \square

13 Corollary. *If \mathcal{M} is a Borel class of models such that $\cong_{\mathcal{M}}$, the isomorphism relation on \mathcal{M} is Borel, then $\cong_{\mathcal{M}}$ is Borel reducible to $E_{\mu\text{-cub}}^\kappa$ for all regular $\mu < \kappa$.*

Proof. Using similar techniques as in classical descriptive set theory (see e.g. [Gao09, Lemma 12.2.7]) one can show that a Borel isomorphism can be reduced to an iterated jump of identity. \square

14 Corollary. *Suppose T a countable complete first-order classifiable (super-stable with NDOP and NOTOP) and shallow theory, then $\cong_T^\kappa \leq_B E_{\mu\text{-cub}}^\kappa$.*

Proof. By [FHK13, Theorem 68] the isomorphism relation of a classifiable shallow theory is Borel, so we apply Corollary 13. \square

We have shown in [FHK13, Theorem 75] that under certain cardinality assumptions on κ , a complete countable first-order theory T is classifiable if and only if for all regular $\mu < \kappa$, $E_{\mu\text{-cub}}^2 \not\leq_B \cong_T$, see Definition 7, where \cong_T is the isomorphism on $\text{Mod}(T)$. Clearly $E_{\mu\text{-cub}}^2 \leq_B E_{\mu\text{-cub}}^\kappa$.

15 Question. Is $E_{\mu\text{-cub}}^\kappa$ reducible to $E_{\mu\text{-cub}}^2$?

If the answer to Question 15 is “yes” then using [FHK13, Theorem 75] we obtain: Suppose T_1 and T_2 are complete first-order theories with T_1 classifiable and shallow and T_2 non-classifiable. Also suppose that $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$. Then \cong_{T_1} is Borel reducible to \cong_{T_2} .

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